Section 2.8

The Intermediate Value Theorem

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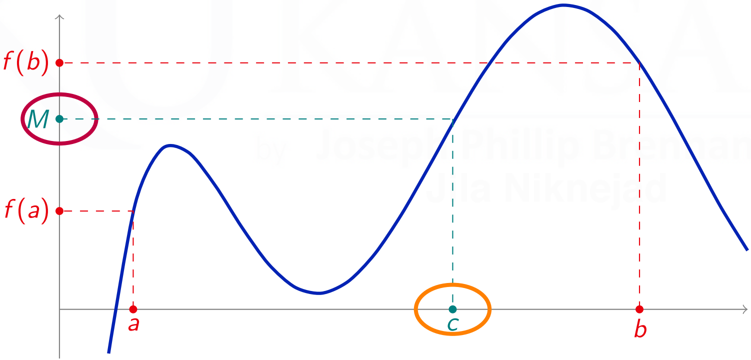
(1) Intermediate Value Theorem

(2) Estimating Zeros of a Function by Bisection

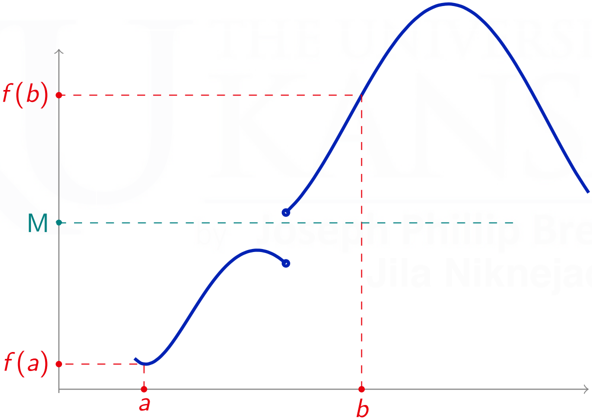
The Intermediate Value Theorem

If f is continuous on the interval $[a, b]$, then for every value M between $f(a)$ and $f(b)$, there exists at least one value c in (a, b) such that

$$f(c) = M$$



Intermediate Value Theorem is true if f is continuous on $[a, b]$.



Estimating Zeros by Bisection

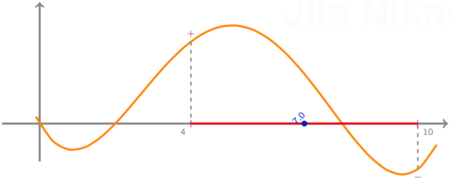
Existence of Zeros

If f is continuous on $[a, b]$ and if $f(a)$ and $f(b)$ are nonzero and have opposite signs, then f has a zero in (a, b).

This rule is a consequence of the Intermediate Value Theorem.

We can use this rule to approximate zeros, by repeatedly bisecting the interval (cutting it in half).

Each time we bisect, we check the sign of $f(x)$ at the midpoint to decide which half to look at next.



Example I: Show that $f(x) = \cos^2(x) - 2\sin\left(\frac{x}{4}\right)$ has a zero in $(0, 2)$. Then locate the zero more accurately using bisection.

Solution: Observe that f is continuous, $f(0) = 1$ is positive, and $f(2) \approx -0.8$ is negative. Therefore, f has a zero in the interval $(0, 2)$.

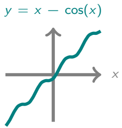
To locate a zero, we repeatedly split intervals in half and apply the Intermediate Value Theorem to each half:

Interval	**Midpoint**	**Function Values**	**Conclusion**
$[\underset{+}{0}, \underset{-}{2}]$	1	$f(1) \approx -0.2$	Zero in $(0, 1)$
$[\underset{+}{0}, \underset{-}{1}]$	0.5	$f(0.5) \approx 0.5$	Zero in $(0.5, 1)$
$[\underset{+}{0.5}, \underset{-}{1}]$	0.75	$f(0.75) \approx 0.2$	Zero in $(0.75, 1)$
$[\underset{+}{0.75}, \underset{-}{1}]$	0.875	$f(0.875) \approx -0.02$	Zero in $(0.75, 0.875)$
$[\underset{+}{0.75}, \underset{-}{0.875}]$	0.8125	$f(0.8125) \approx 0.07$	Zero in $(\underset{+}{0.8125}, \underset{-}{0.875})$

The interval is accurate up to one decimal place.

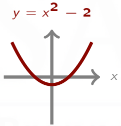
Example II: Does $\cos(x) = x$ have a solution?

The Intermediate Value Theorem applies to $f(x) = x - \cos(x)$ on any interval. Since $f(0)$ is negative and $f\left(\frac{\pi}{2}\right)$ is positive, there is a value c in $\left(0, \frac{\pi}{2}\right)$ where $c - \cos(c) = 0$.



Example III: Show that $\sqrt{2}$ exists.

The Intermediate Value Theorem applies to $f(x) = x^2 - 2$. There is a value c in $(1, 2)$ where $c^2 - 2 = 0$. Since $c^2 = 2$, $\sqrt{2}$ exists and equals c.



Example IV: Does $f(x) = x^4 + x - 4$ have a zero?

Solutions exist in the intervals $(-2, -1)$ and $(1, 2)$. Does $f(x)$ have only two zeros?